ROMA TRE
UNIVERSITÀ DEGLI STUDI

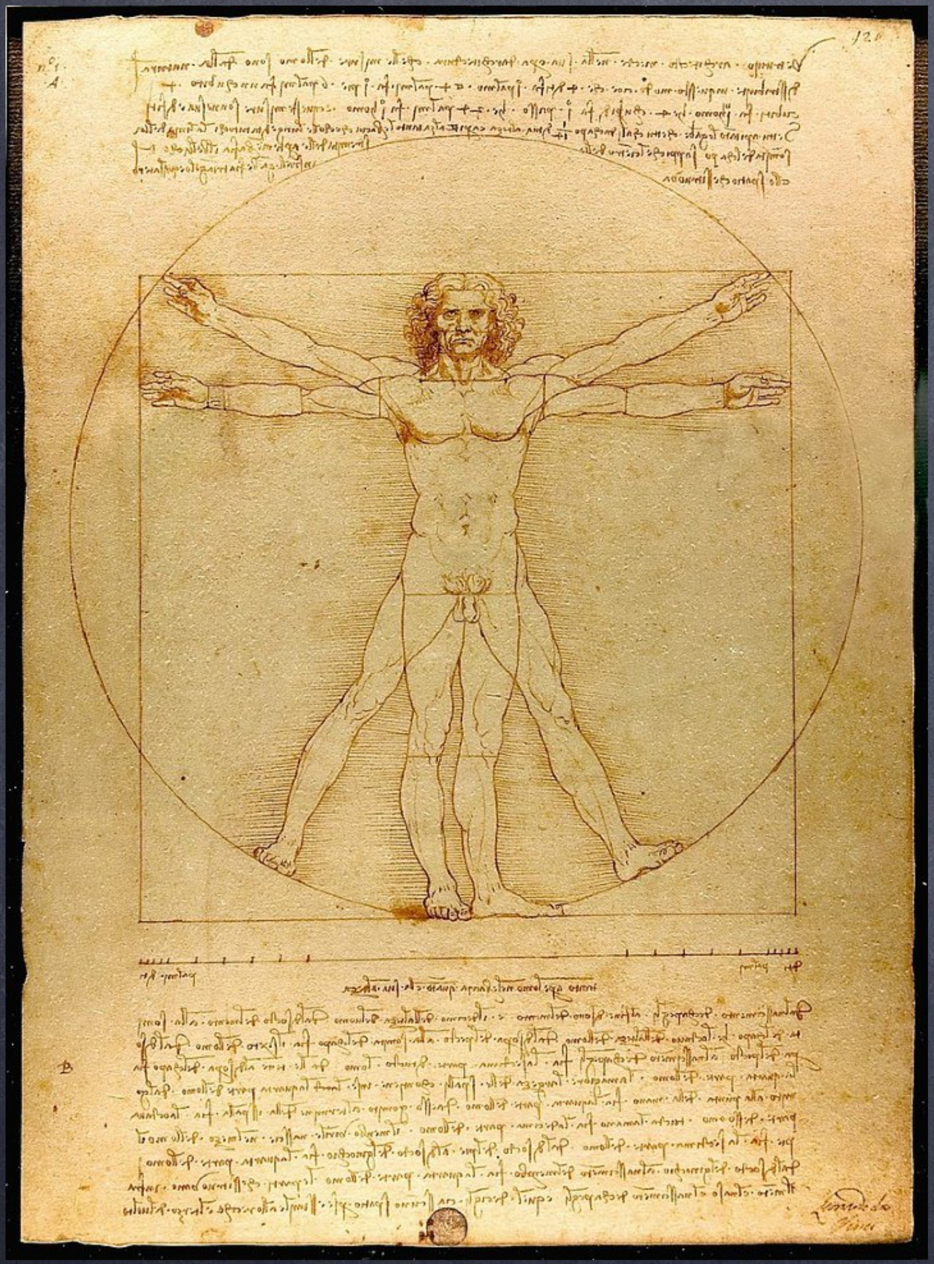
CNR-INO

QCUMbER

A primer on (quantum) metrology

NEw Quantum Optics group
Dipartimento di Scienze
Università Roma Tre, Italy

QCUMbER Conference, Oxford July 8th-13th 2018



Metrology (n): the scientific study of measurement. From Greek metron

Omne trinum perfectum

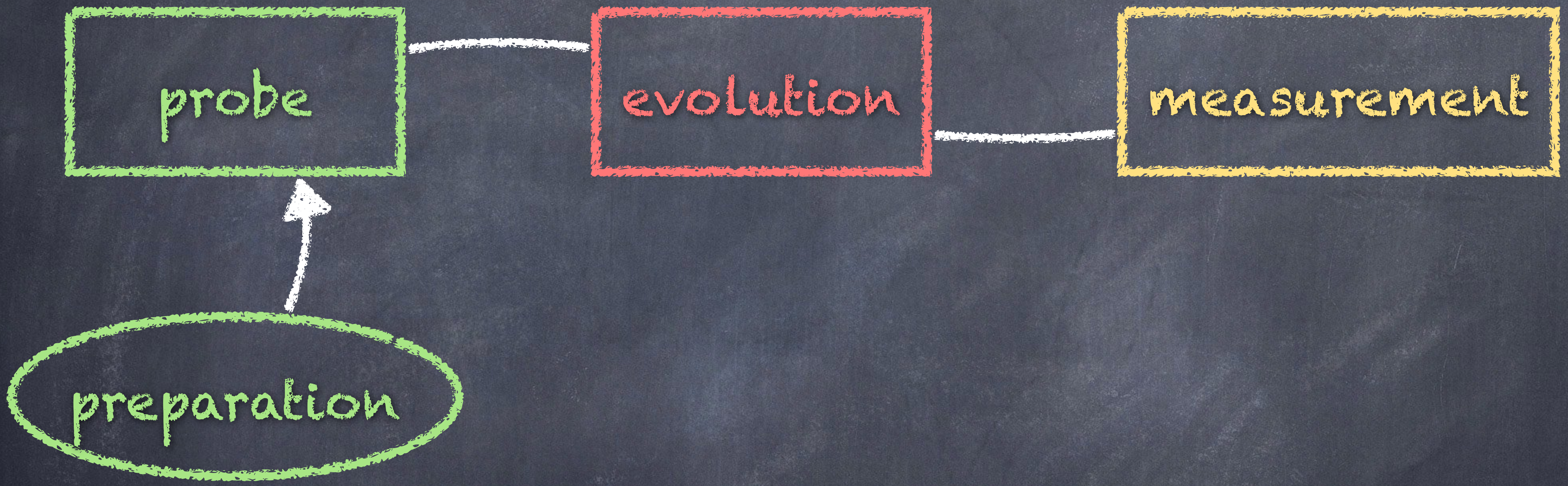
Omne trinum perfectum

probe

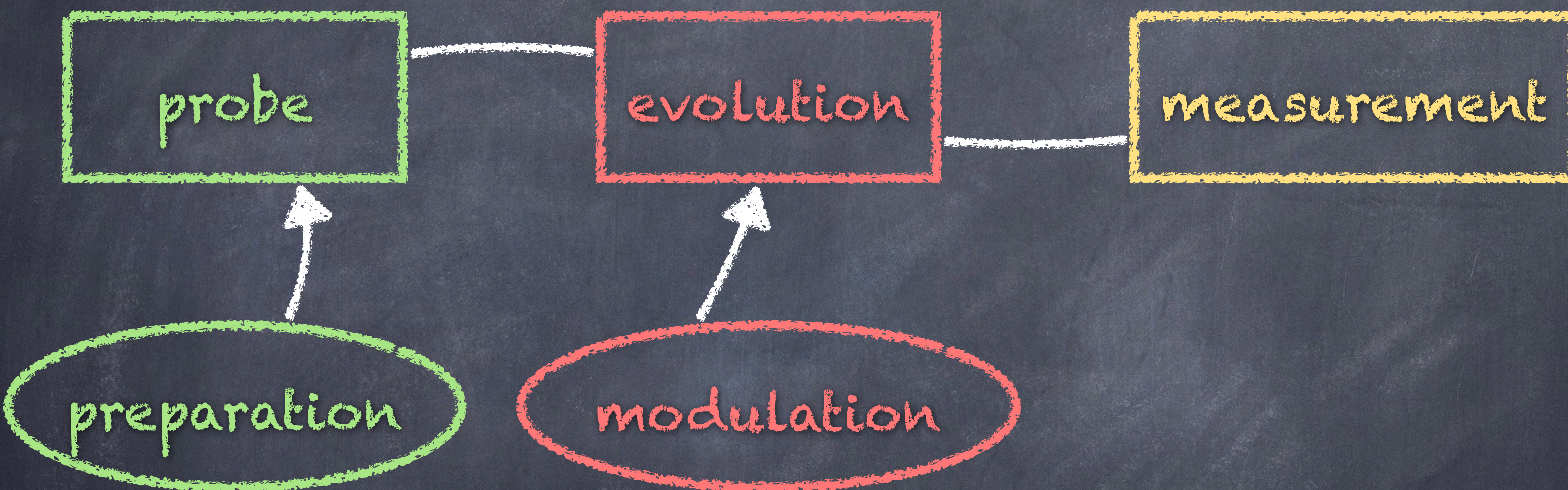
Omne trinum perfectum

Omne trinum perfectum

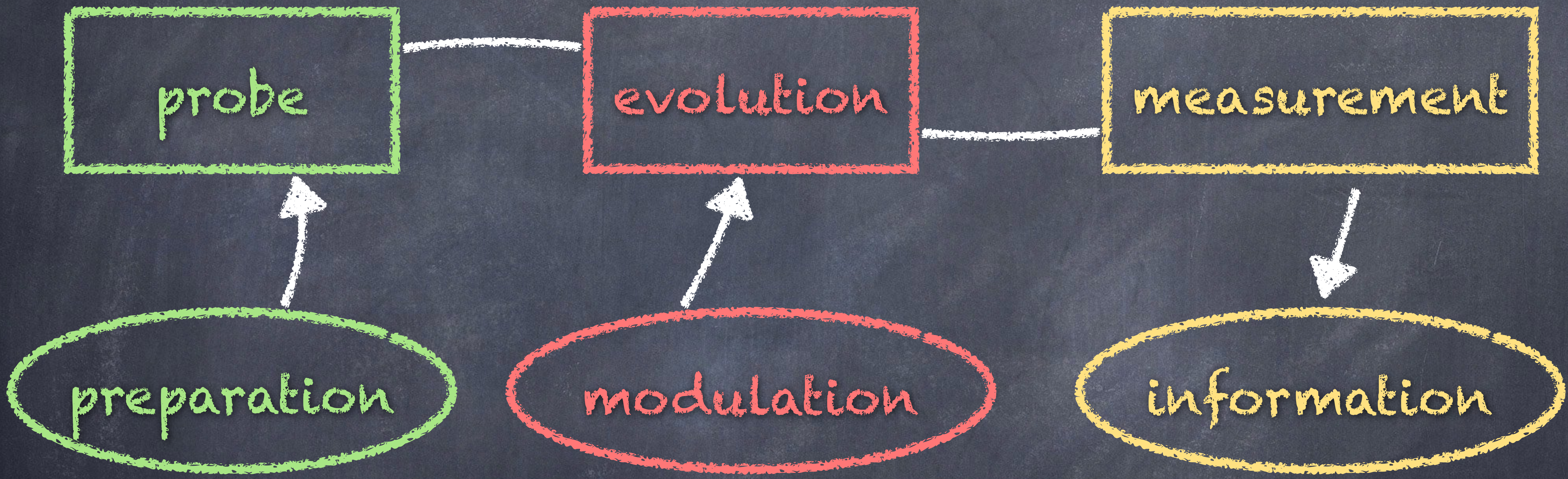
Omne trinum perfectum



Omne trinum perfectum



Omne trinum perfectum



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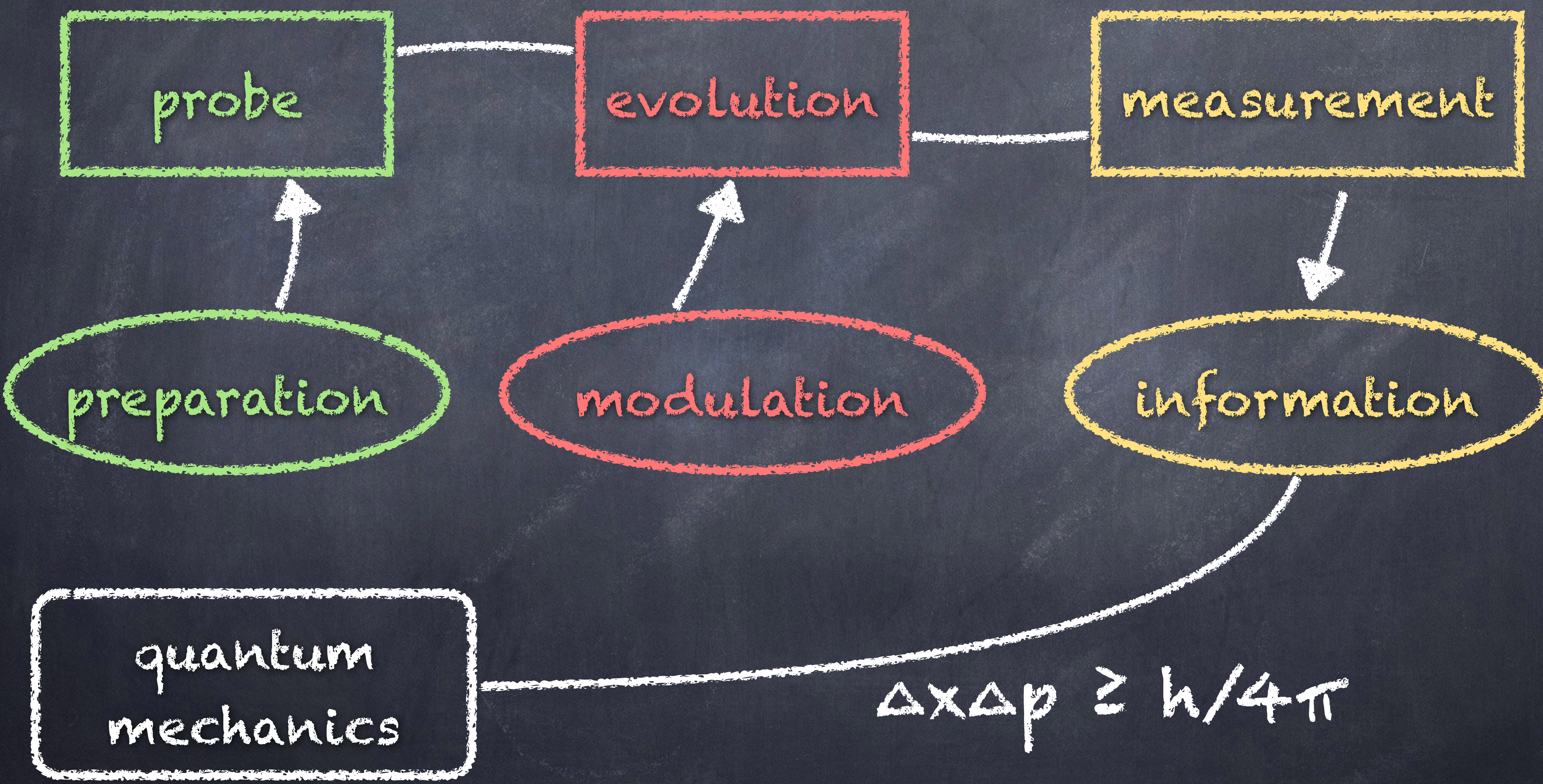
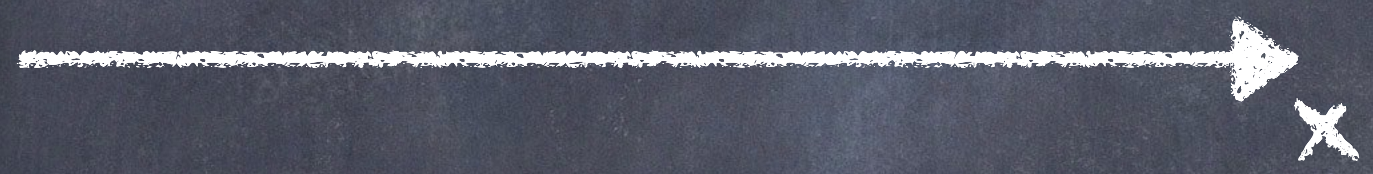
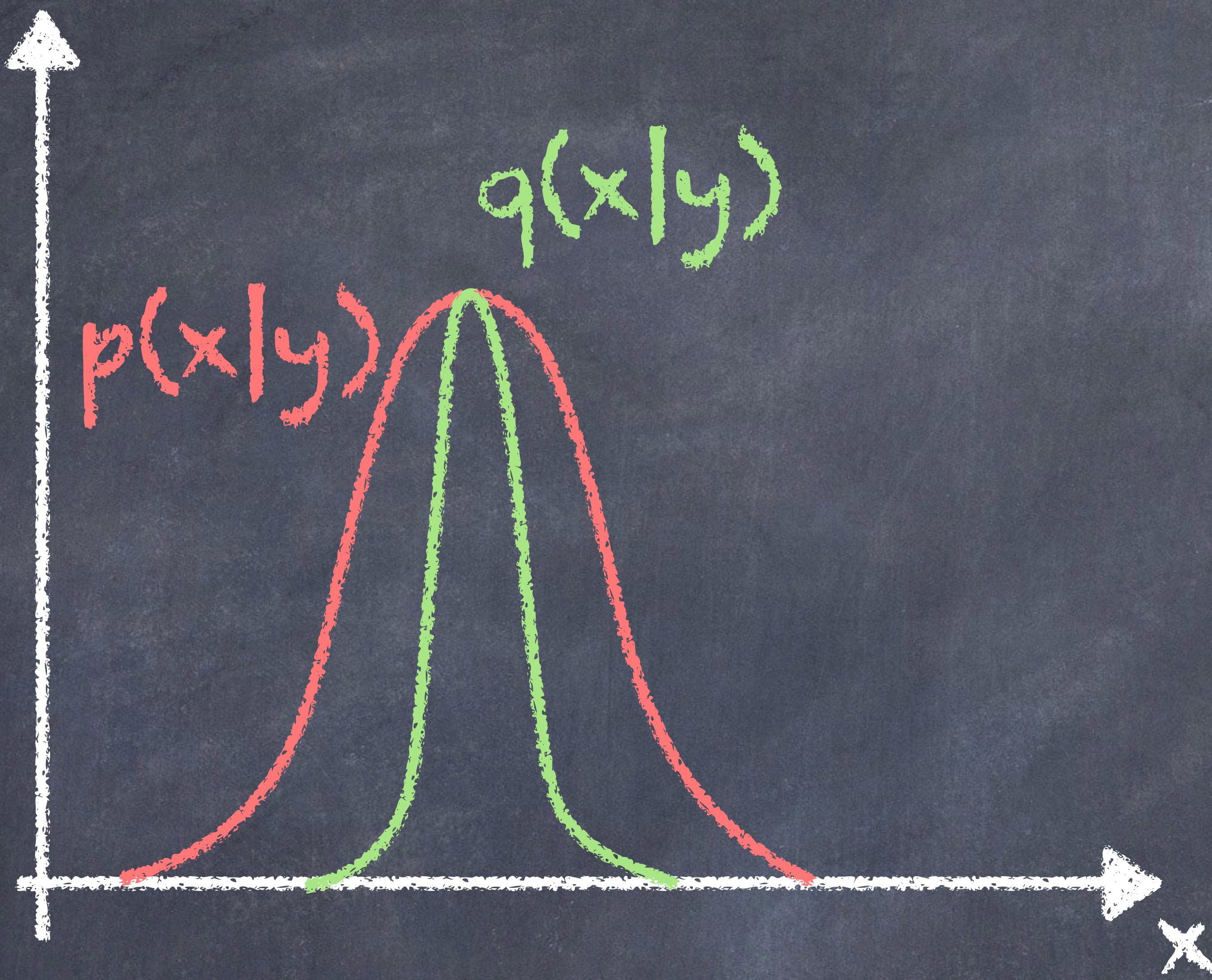
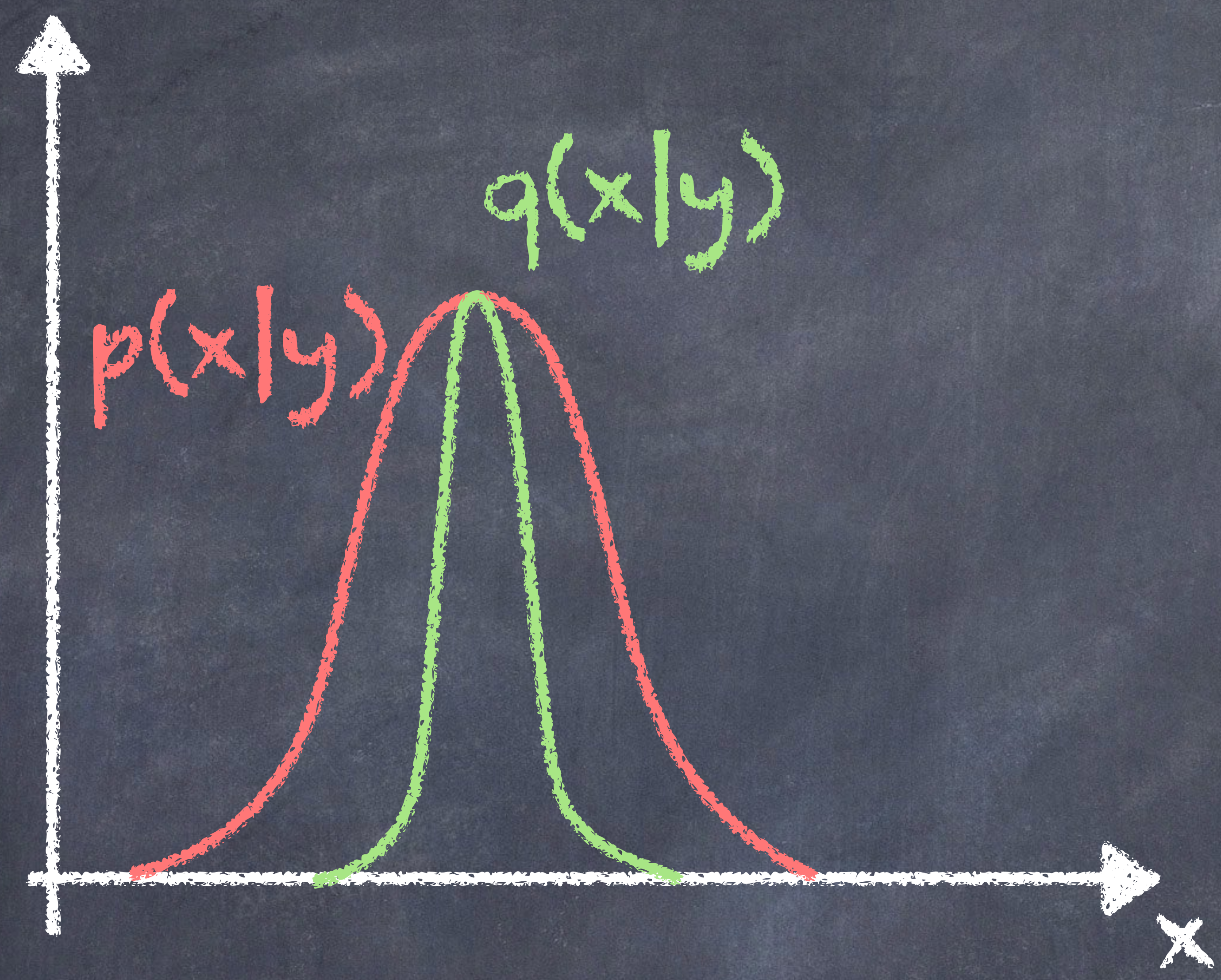



x


p(x|y)
x

p(x|y)
p(x|y')
x

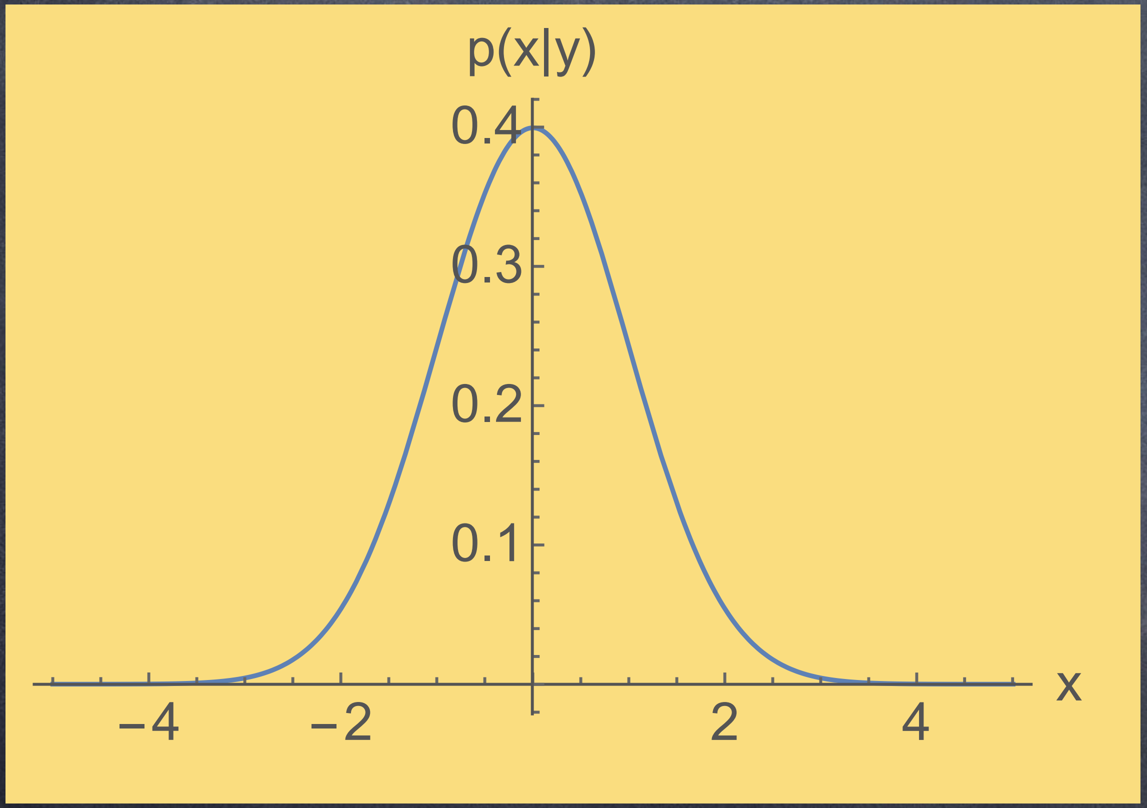

q(x|y)
p(x|y)
x



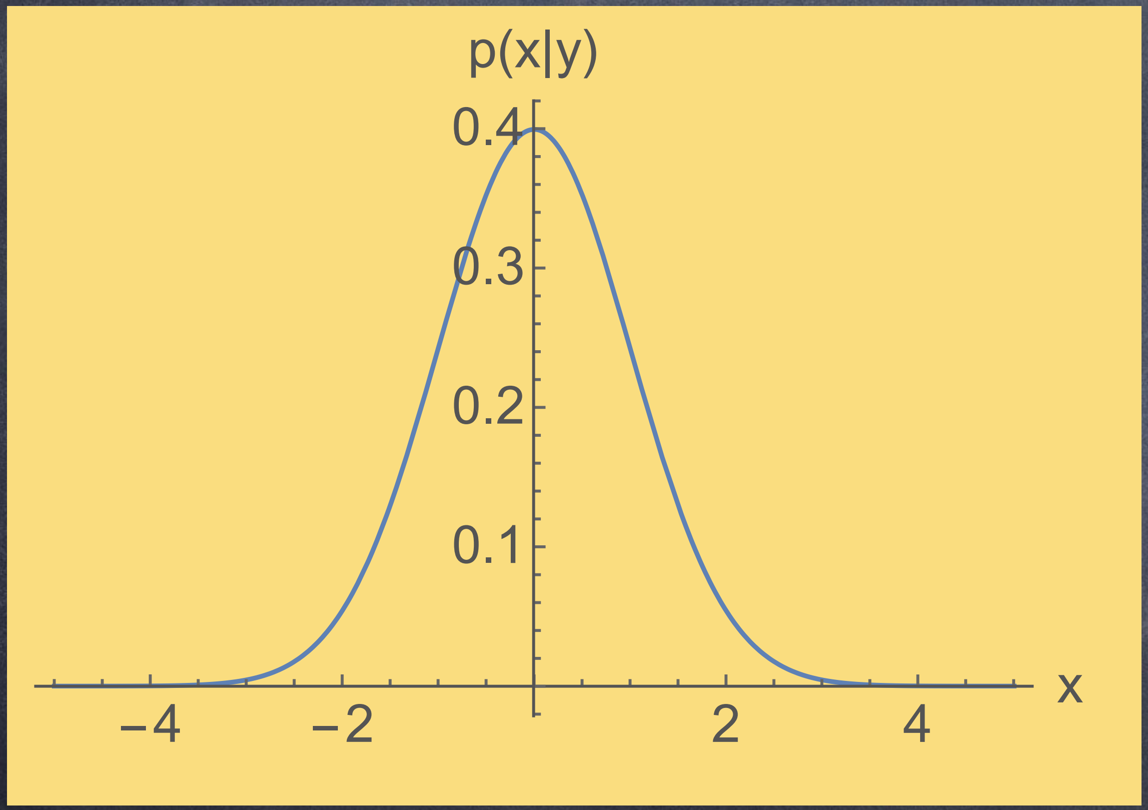
q(x|y) is better than p(x|y)

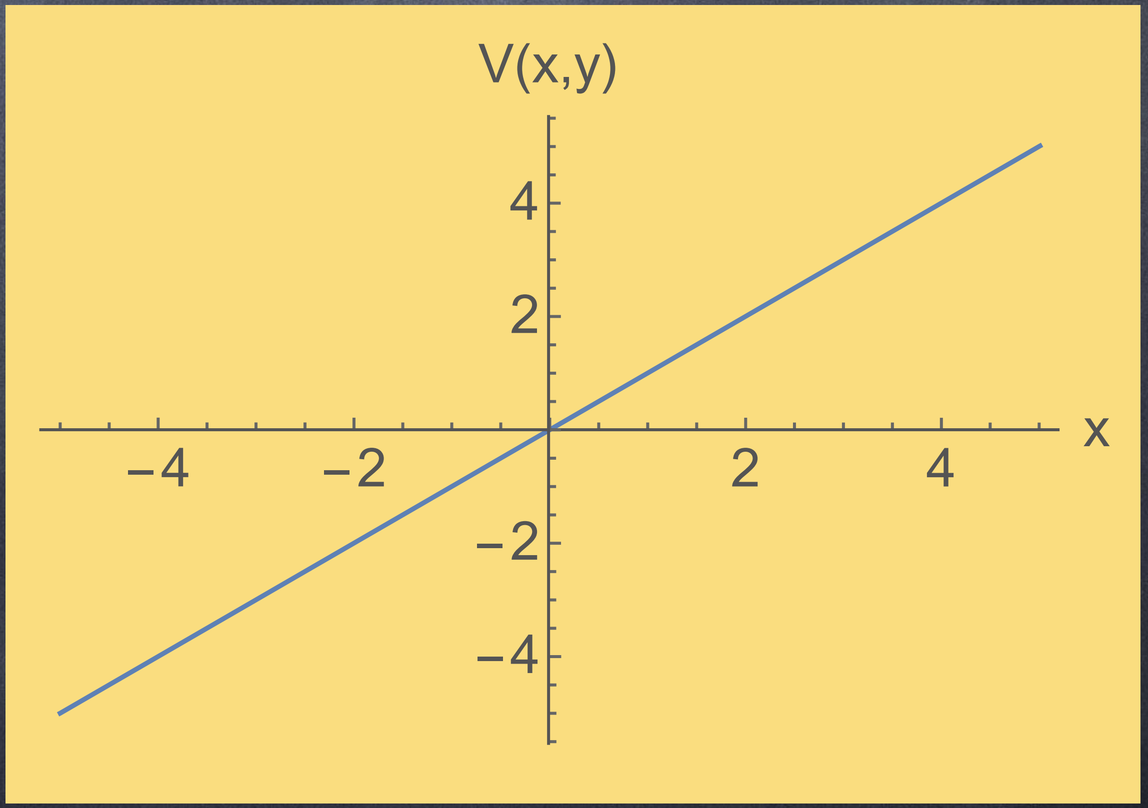
score function

score function



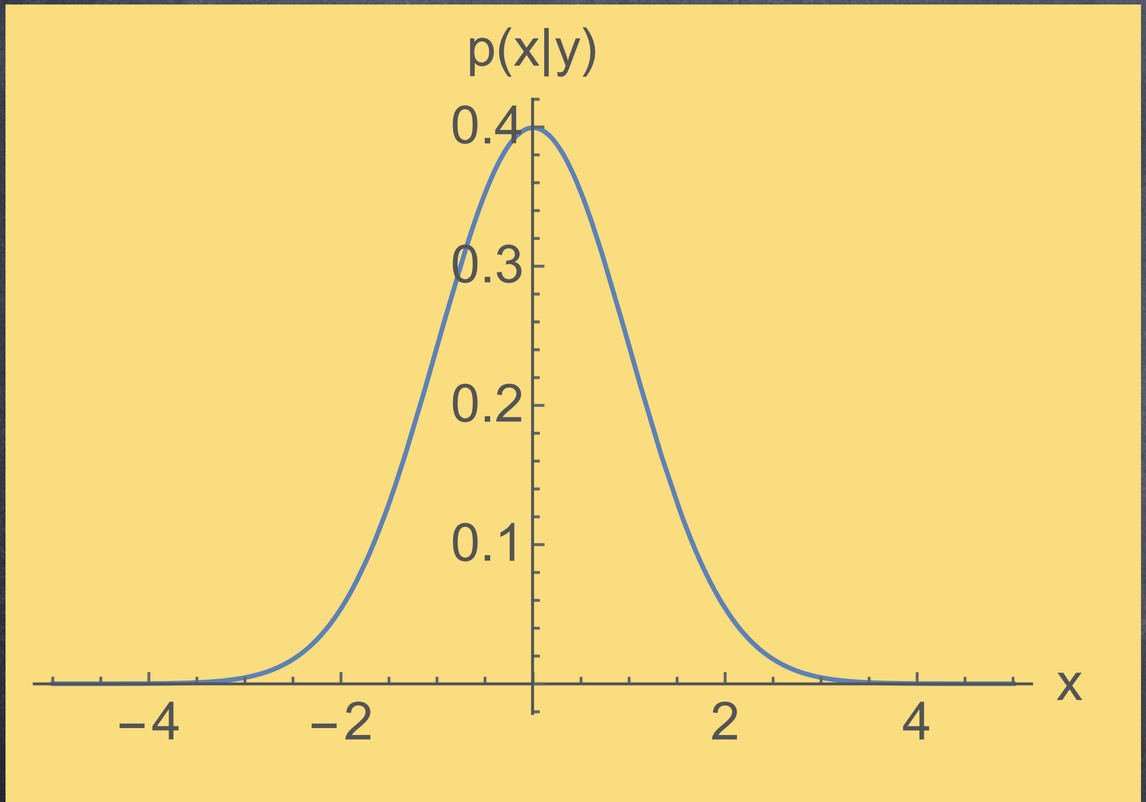
score function

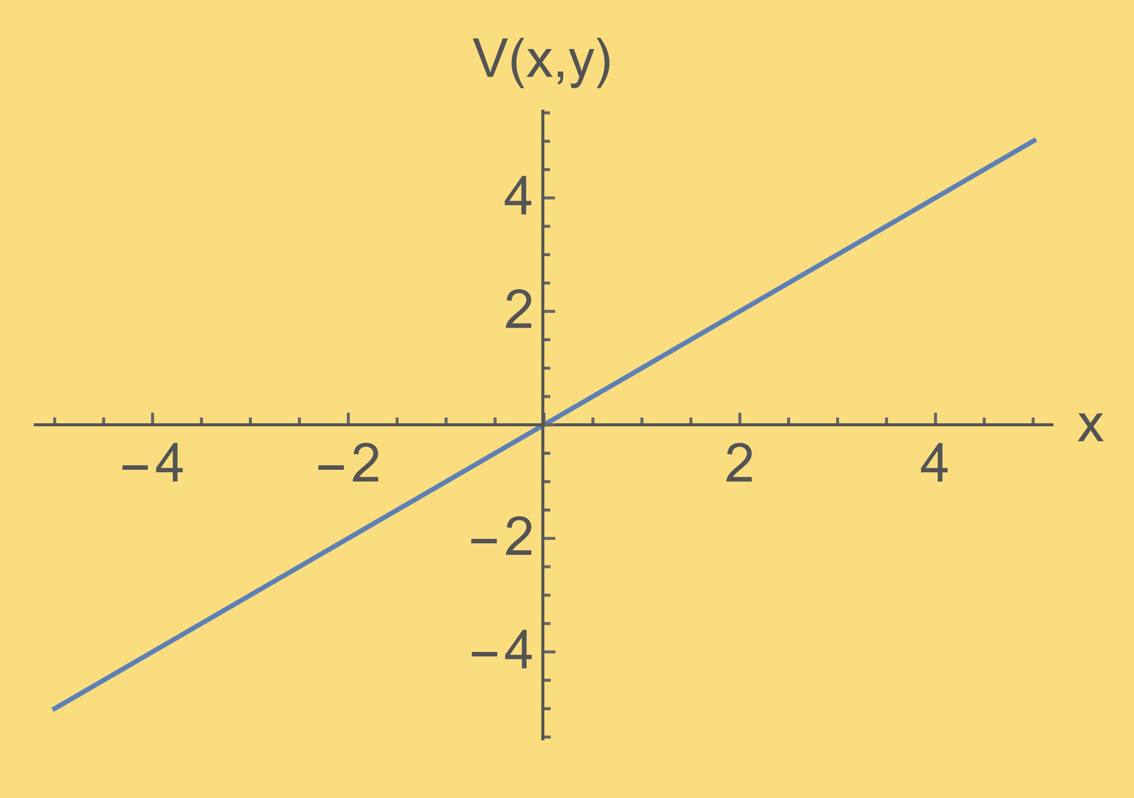




score function

$$V(x,y) = \partial_y log(p(x,y)) = \frac{\partial_y p(x,y)}{p(x,y)}$$





Fisher information

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$$\mathcal{I}(y) = \mathbf{E}[V(x,y)^2] = \int \frac{(\partial_y p(x,y))^2}{p(x,y)}\, dx$$

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for a Gaussian pdf

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$$\mathcal{I}(y) = \frac{1}{\sigma^2}$$

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$$\mathcal{I}(y) \geq \frac{1}{\mathrm{Var}[y]}$$

Cramér-Rao bound

Fisher information

$$\mathcal{I}(y) = \mathbf{E}[V(x,y)^2] = \int \frac{(\partial_y p(x,y))^2}{p(x,y)}\,dx$$

[lots of maths which is essentially a clever use of the Cauchy Schwartz inequality]

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Cramér-Rao bound

what if we cumulate M experiments

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$$\mathcal{I}_{\text{tot}}(y) \geq M\mathcal{I}(y)$$

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Cramér-Rao bound $\mathcal{I}(y) \geq \frac{1}{M \mathrm{Var}[y]}$

1. valid for unbiased estimators

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$$\mathbf{E}[y_{\mathrm{est}} - y] = 0$$

Cramér-Rao bound $\mathcal{I}(y) \geq \frac{1}{M \mathrm{Var}[y]}$

1. valid for unbiased estimators

$x_1, x_2, x_3, ..., x_M$ $\quad$ y_{est}

$$\mathbf{E}[y_{\text{est}} - y] = 0$$

$$\mathrm{Var}[y] = \mathbf{E}[(y_{\text{est}} - y)^2]$$

Cramér-Rao bound

$$\mathcal{I}(y) \geq \frac{1}{M\mathrm{Var}[y]}$$

2. valid for local estimation

$x_1, x_2, x_3, \ldots, x_M$ → y_{est}

$$\mathbf{E}[y_{\mathrm{est}} - y] = 0$$

$$\mathrm{Var}[y] = \mathbf{E}[(y_{\mathrm{est}} - y)^2]$$

Cramér-Rao bound $\mathcal{I}(y) \geq \frac{1}{M\text{Var}[y]}$

3. valid asymptotically

$x_1, x_2, x_3, ..., x_M$ $\quad\quad$ y_{est}

$$\mathbf{E}[y_{\text{est}} - y] = 0$$

$$\text{Var}[y] = \mathbf{E}[(y_{\text{est}} - y)^2]$$

Cramér-Rao bound $\mathcal{I}(y) \geq \frac{1}{M\mathrm{Var}[y]}$

3. valid asymptotically

$x_1, x_2, x_3, ..., x_M$ → y_{est}

M large

$$\mathbf{E}[y_{\mathrm{est}} - y] = 0$$

$$\mathrm{Var}[y] = \mathbf{E}[(y_{\mathrm{est}} - y)^2]$$

Where do we get $p(x|y)$ from?

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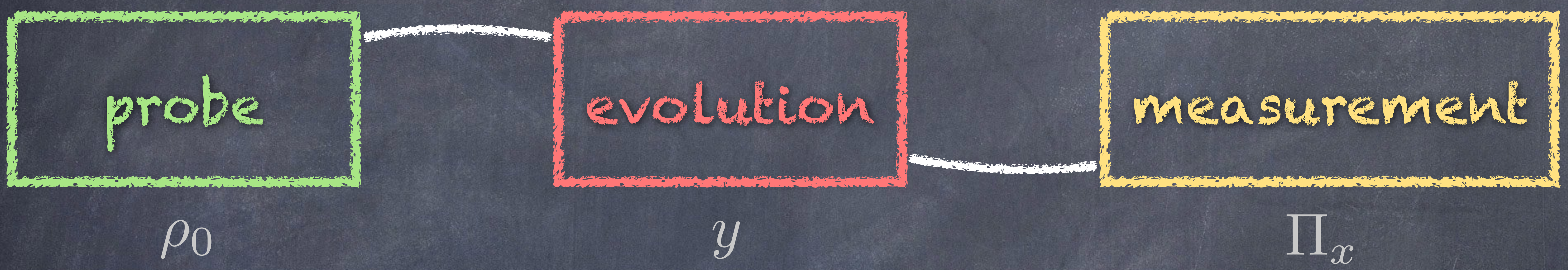

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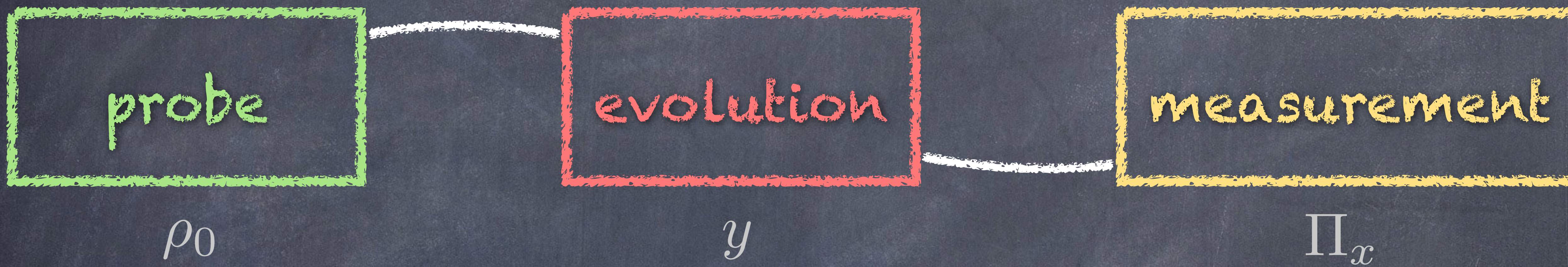

$$p(x|y) = \mathrm{Tr}[\rho_y \Pi_x]$$

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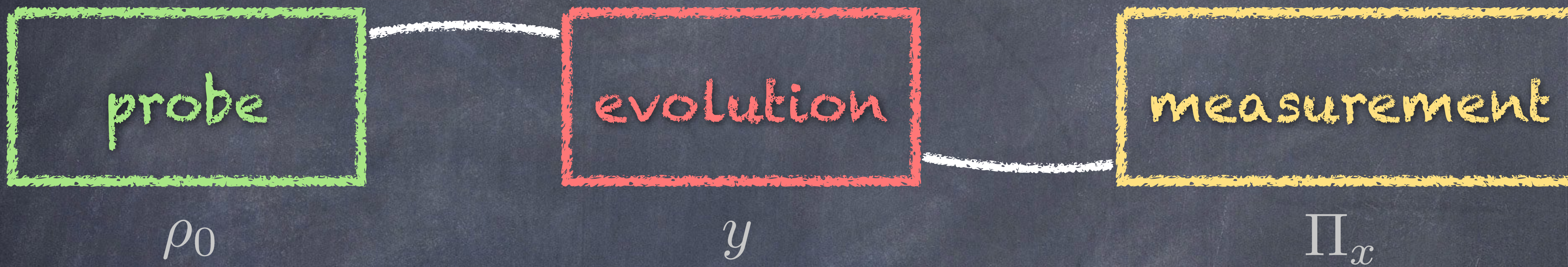

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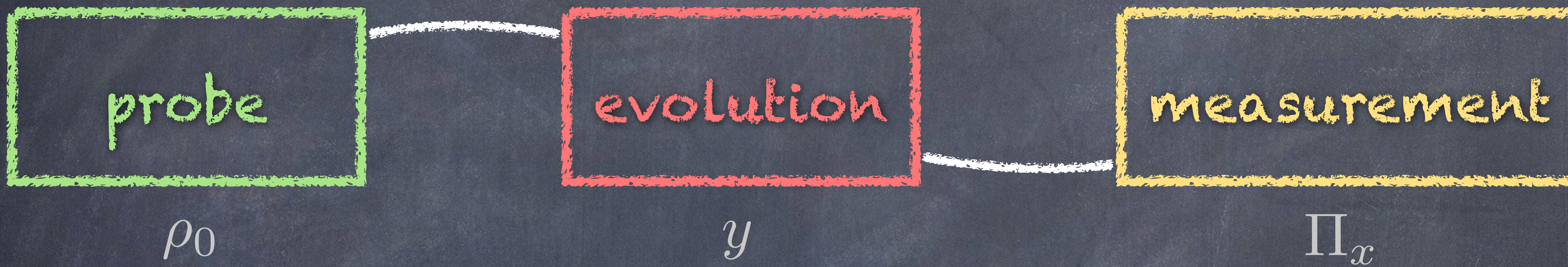
- which POVM?

Where do we get p(x|y) from?



$$p(x|y) = \mathrm{Tr}[\rho_y \Pi_x]$$

- which POVM?
  - which state?

measure in the direction in which the state curves the most

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$$2\partial_y \rho_y = L_y \rho_y + \rho_y L_y$$

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L_y Symmetric Logarithmic Derivative

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$\mathcal{I}_Q(y) \geq \mathcal{I}(y)$ Quantum Cramér-Rao bound

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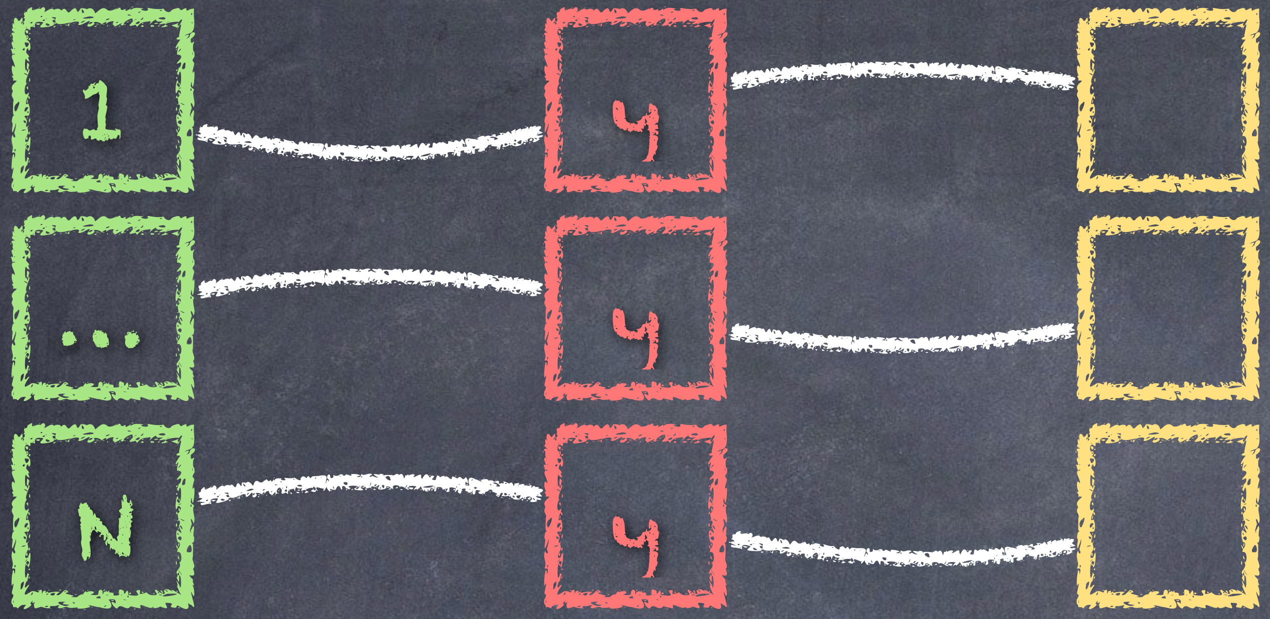
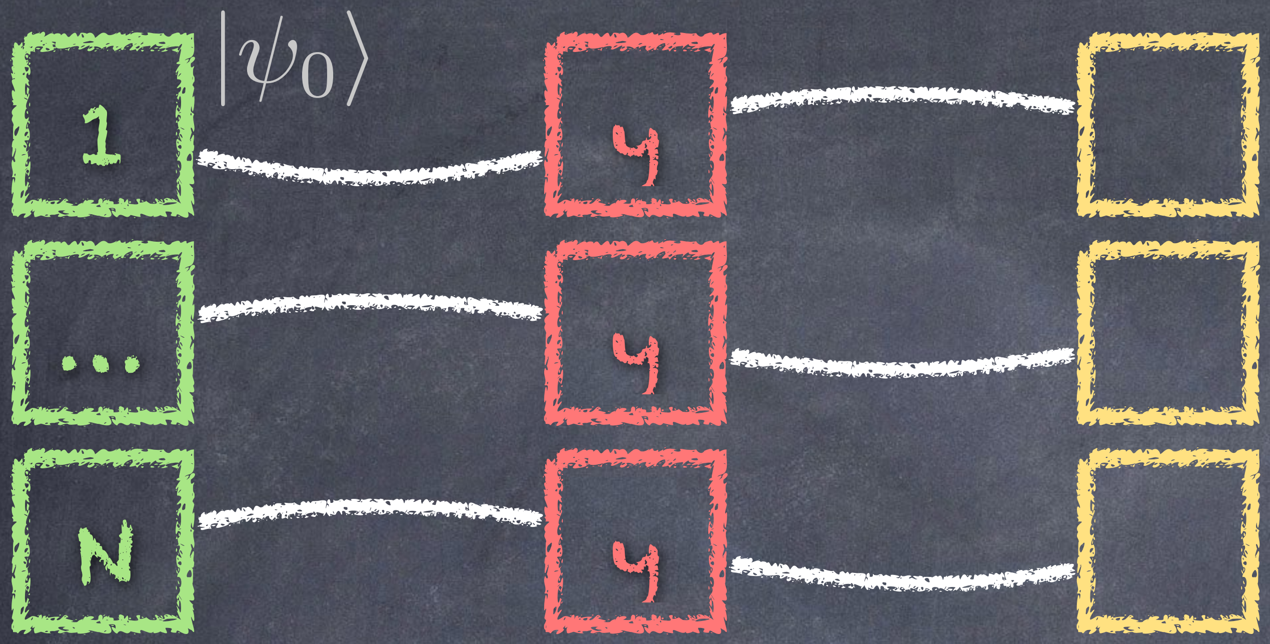
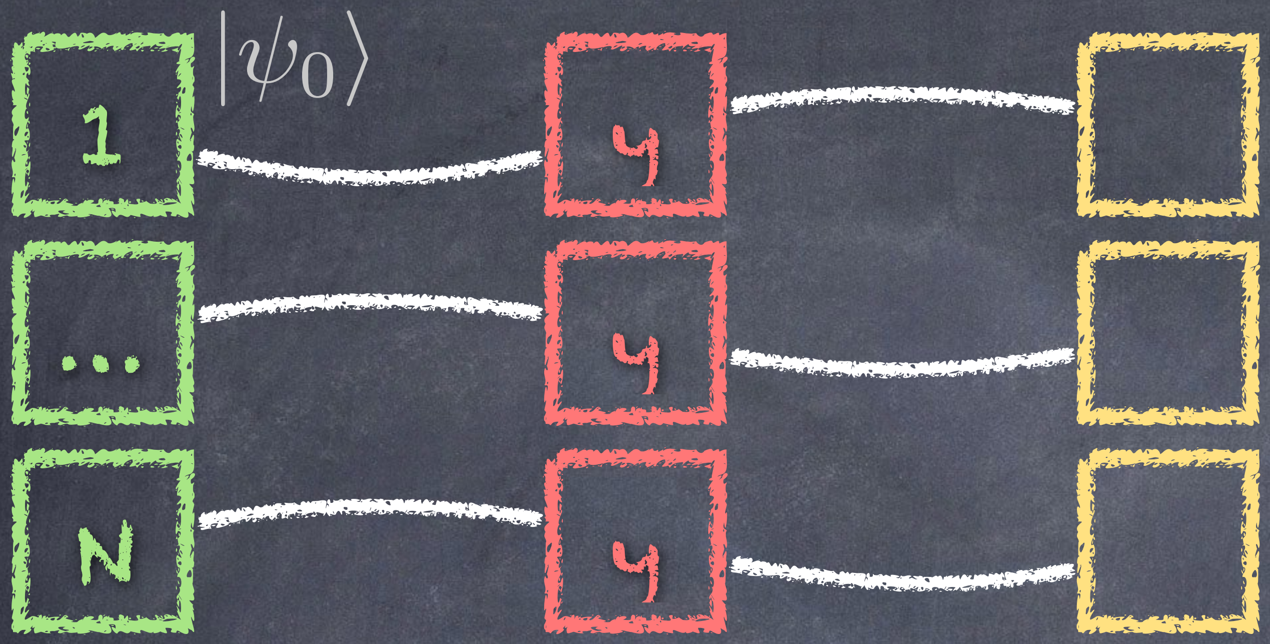
The optimal measurement is the SLD

$$\mathcal{I}_Q(y) = \frac{1}{M\mathrm{Var}[y]}$$


1
y
y
...
y
N

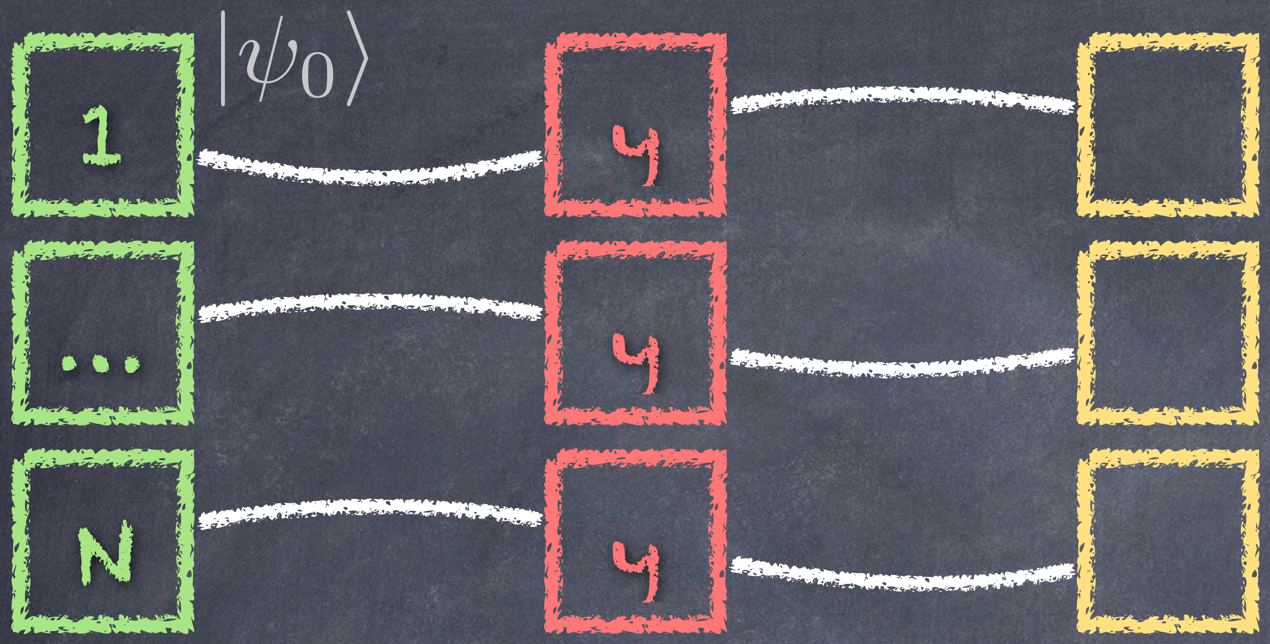


$$U_y = \exp(-iHy)$$



$\mathcal{I}_Q(y) \sim N$

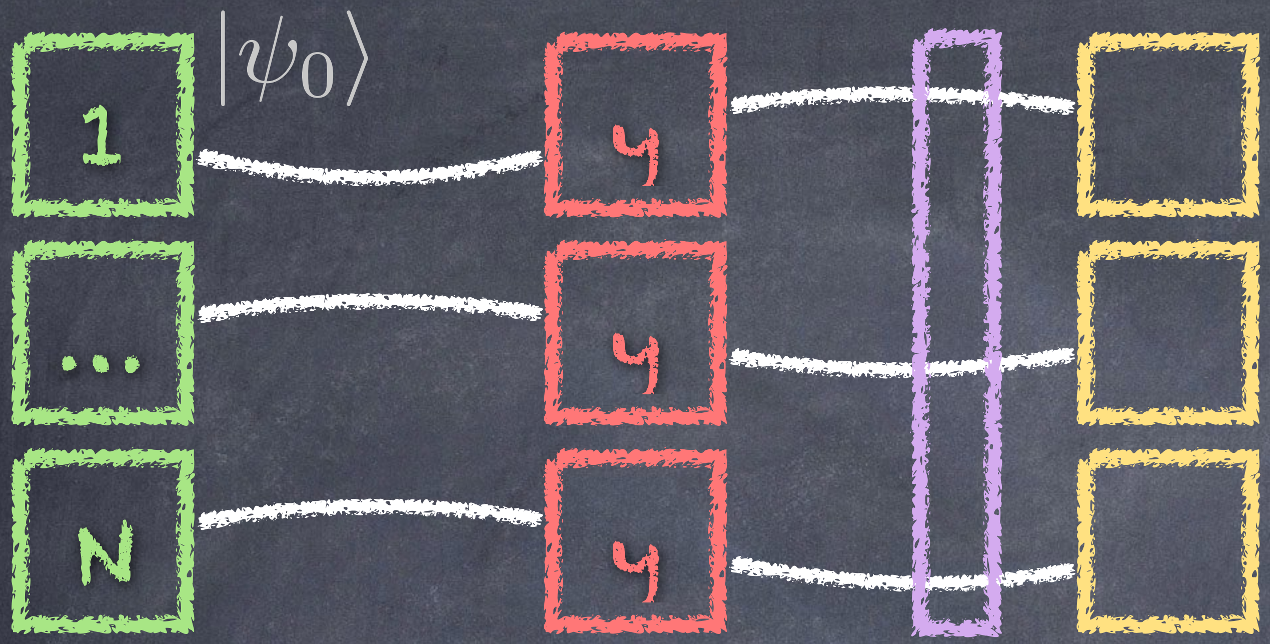
$U_y = \exp(-iHy)$



$$U_y = \exp(-iHy)$$

c

c $\mathcal{I}_Q(y) \sim N$

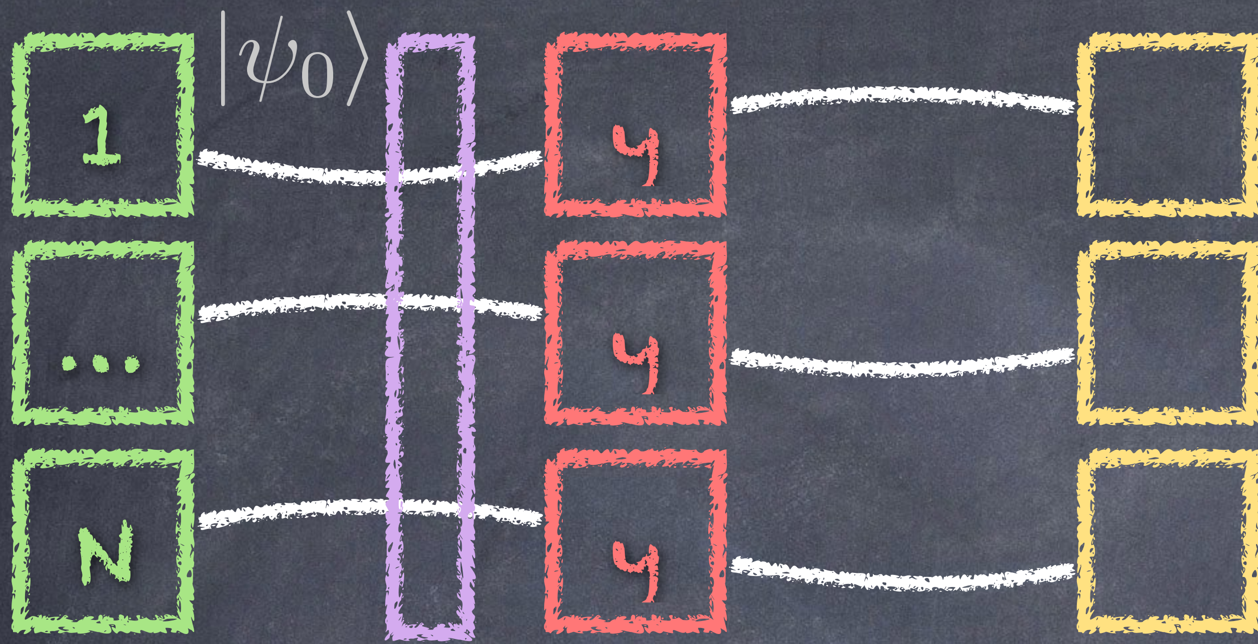


$$U_y = \exp(-iHy)$$

C

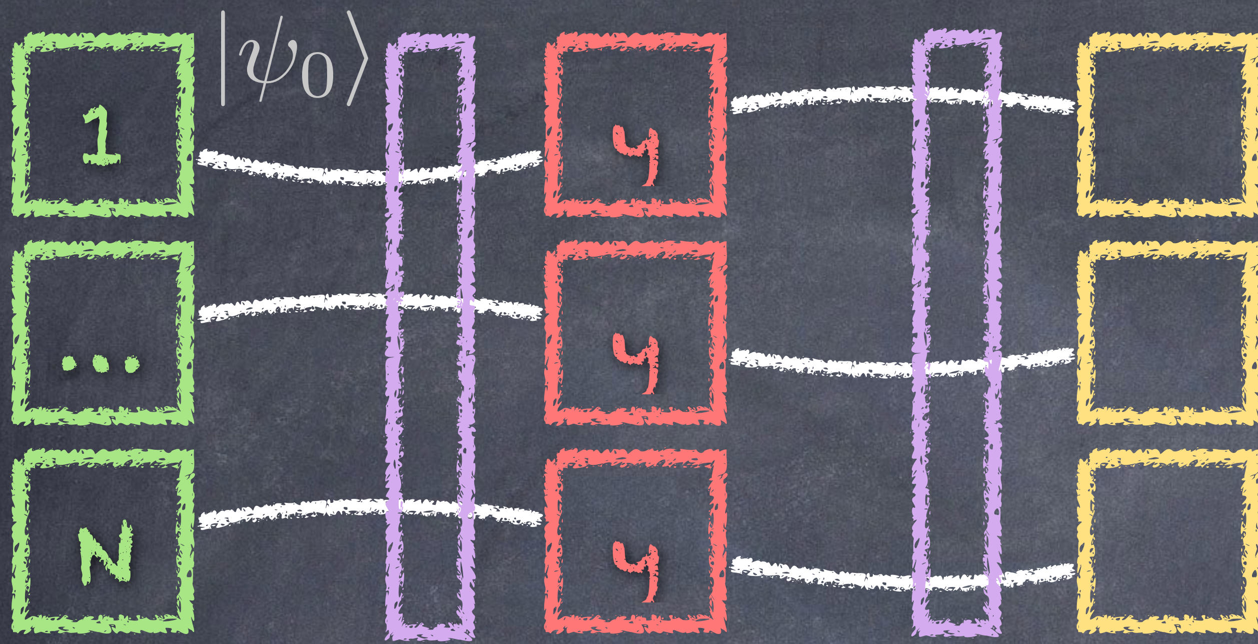
C $\mathcal{I}_Q(y) \sim N$

Q $\mathcal{I}_Q(y) \sim N$



$$U_y = \exp(-iHy)$$

	C	Q
C	$\mathcal{I}_Q(y) \sim N$	$\mathcal{I}_Q(y) \sim N^2$
Q	$\mathcal{I}_Q(y) \sim N$	



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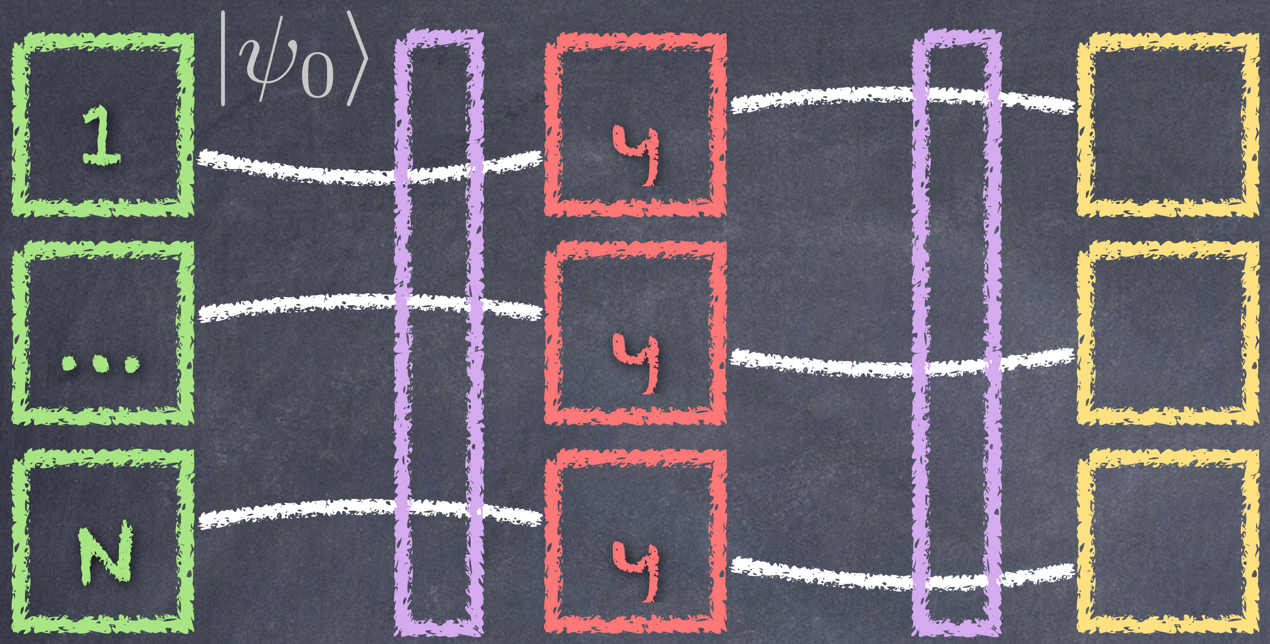

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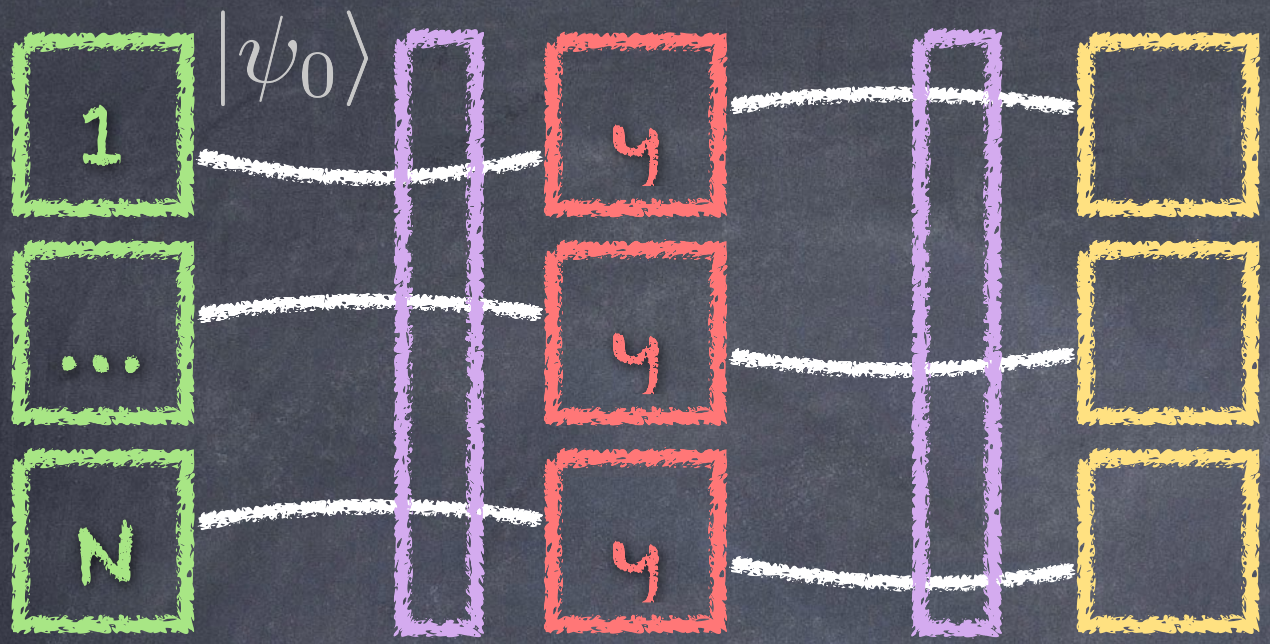

$$U_y = \exp(-iHy)$$

$$\mathrm{Var}[y]\,\Delta^2 H_{\mathrm{tot}} \geq \frac{1}{4M}$$

generalised Heisenberg relation